Optimal Control Problems and Riccati Differential Equations

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Abstract: The paper deals with linear quadratic (LQ) optimal problems with free and fixed-end point. A unified approach for both problems (with fixed and free end-point) allows establishing of analytical formulae for the solution to the same Riccati equation, with different terminal conditions. Certain procedures which start from an initial adequate condition are established for the problem with free end-point. The necessity of the inverse time computing of the solution to Riccati equation is thus avoided.

Keywords: optimal control, Riccati equations, differential equations, analytical solution, final values

1. INTRODUCTION

The Riccati matrix differential equation (RMDE) appears in two main problems regarding the linear systems: the optimal control and the optimal estimation. The RMDEs in these cases can differ by the terminal conditions, which can be done at the initial moment t_0 or at the final moment t_f . The LQ problem with fixed end-point (**P1** problem) leads to a RMDE with fixed initial condition. The most known LQ problem refers to the free end-point case (**P2** problem) and it leads to a RMDE with imposed final value. The first situation is also met in the optimal estimation problems. The paper will deal in the sequel only with the optimal control problem in both mentioned variants. Some results established for the problem with fixed end-point can be adapted for the estimation problems.

The present paper refers to the well known LQ problem, which is treated in very much number of papers or books. We mention here for exemplification (Athans & Falb 1966; Lee & Markus 1967; Anderson & Moore 1990; Abou-Kandil et al. 2003) as fundamental books from different periods, referring to this problem. We refer especially to the Riccati differential equation, which is the key for solving of the mentioned problems. There are different categories of methods for this aim and we can distinguish the following main groups:

- Direct integration of RMDE.

- Iterative solving of a simpler first order equation. We mention in this direction the use of Lyapunov or Bernoulli equation (Kenney & Leipnik 1985), or the Chandrasekhar method (extended for time-variant problems by Lainiotis (1976)).

- Analytical, non recursive procedures: the most variants use a factorization of the solution obtained from the partitioning of a 2nx2n transition matrix. The method was analysed by Davison and Maki (1973) for the time-invariant case and it is presented by Athans and Falb (1966) for a general case. Formulae which use only nxn transition matrices were also proposed (Botan 1985; Rusnak 1988; Botan & Ostafi 2010). In the same category of non recursive solutions one can be

mentioned (Incertis 1983; Choi & Laub 1990) and others.

Certainly, there are different other methods which cannot be included in the above categories (for instance, (Sorine & Winternitz 1985)). A comparison of some methods is given by Kenney and Leipnik (1985). In the last years the researches were oriented to the extension of the possibilities of solving RMDE, to large scale systems, to the problems with difficulties, to different generalized problems and also, different procedures were proposed for problem solving, like those based on new mathematical programming methods (Lee 2005; Barabanov & Ortega 2004; Hansson 2000; Yao, Zhang & Zhou 2001).

All above considerations refer especially to **P2** problems. In (Anderson & Moore, 1990; Abou-Kandil et al. 2003) are also presented important aspects referring to **P1** problems. Some especial aspects referring to Riccati equation in **P1** problems are indicated in (Friedland 1967; Brunovsky &Komornik 1981; Botan, Ostafi & Onea 2003).

The establishing of new basic procedures for RMDE is not a closed problem, because the simplification of the implementation remains a desideratum. We refer, for instance, to the optimal control of the electrical drives, where it is necessary to use sampling periods of milliseconds. Since the complexity of the algorithms for modern drive control method (like vector control) is very high, the introducing of the optimal control is possible only if the corresponding algorithm is simple.

The paper presents certain analytical solutions to RMDE for the optimal problem with fixed end-point (P1) and with free end-point (P2). These methods allow a fast computing and ensure a high accuracy. In addition, an analytical formula for solution to RMDE allows establishing the initial value for this solution in P2 problem and therefore, the problem with final condition for RMDE can be transformed in one with initial condition. In this case, the real time computing is simpler, because the solution at every moment can be established from the results obtained at the previous step. In this way, the computing in inverse time is avoided. A similitude between LQ with free and fixed end-point problems is put in evidence. In fact, the same Riccati equation arises in both cases. The difference consists in terminal (initial or final) condition which holds in every case.

Besides the mentioned theoretical aspects, the proposed methods have an advantage in implementation, because the computing time is reduced in comparison with other procedures. This aspect is important in applications with severe time restrictions.

The next section presents certain common aspects for both mentioned problems. The main results of the paper referring to the solution to RMDE are developed in the sections 3. Simulation results and conclusions end the paper.

2. PROBLEMS FORMULATION AND OPTIMALITY CONDITIONS

The LQ optimal problems refer to a linear system and a quadratic criterion. Usually LQ denomination is adopted for the problems with free end-point, but the problems with fixed end-point can be considered in the same category, since they also refer to a linear system and a quadratic cost function. Only the problems with finite final time will be considered. The paper deals with time invariant systems and only some remarks regarding the extension to time variant case will be presented.

A linear time-invariant system is considered:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \ \mathbf{x} \in \mathbf{R}^n, \ \mathbf{u} \in \mathbf{R}^m.$$
(1)

The general form of the considered criterion is

$$J = \frac{1}{2}x^{T}(t_{f})Sx(t_{f}) + \frac{1}{2}\int_{t_{0}}^{t_{f}} \left[x^{T}(t)Qx(t) + u^{T}(t)Pu(t)\right]dt$$
(2)

(T denotes the transposition). The matrices A, B, $S \ge 0, Q \ge 0, P > 0$ are with appropriate dimensions.

The initial moment t_0 and the initial state $x(t_0) = x^0$ are fixed. The following problems are formulated:

P1 (with fixed end-point): find the optimal control u(t) which transfer the system (1) from x^0 in the final state $x(t_f) = x^f = 0$ (t_f is fixed) so that to be minimized the performance index (2), with S = 0.

P2 (with free end-point): find the optimal control u(t) which transfer the system (1) from x^0 in a free point (t_f is fixed) so that to be minimized the performance index (2).

The problems **P1** and **P2** will be studied using a common procedure, and the attention will be focused on RMDE which appears in both cases. There are frequently used procedures for **P1** problems that do not involve Riccati equation. Unlike these techniques, one will indicate that it is useful to use this equation in such problems and there is a similitude with **P2** problems.

The necessary optimality condition (Athans & Falb 1966; Anderson & Moore 1990) leads to

$$u(t) = -P^{-1}(t)B(t)\lambda(t), \qquad (3)$$

where $\lambda(t) \in \mathbf{R}^n$ is the co-state vector.

If it is desired to obtain a feedback optimal control, the co-state vector $\lambda(t)$ from (4) is expressed as a function of the state vector

$$\lambda(t) = \tilde{R}(t)x(t), \ \tilde{R}(t) \in \mathbf{R}^{n \times n}$$
(4)

and than the optimal control becomes

$$u(t) = -P^{-1}B^T \hat{R}(t)x(t) .$$
 (5)

The matrix $\tilde{R}(t)$ in (4) and (5) is positive defined and satisfies the well known Riccati matrix differential equation (RMDE) (Athans & Falb 1966)

$$\tilde{R}(t) = \tilde{R}(t)N\tilde{R}(t) - A^T\tilde{R}(t) - A\tilde{R}(t) - Q, \ N = BP^{-1}B^T.(6)$$

The state and co-state variables satisfy the Hamilton canonical equations; taking into account (3), these equations can be written in the form

$$\dot{\gamma}(t) = G\gamma(t) , \qquad (7)$$

where

$$\gamma(t) = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \in \mathbf{R}^{2n}, \ G = \begin{bmatrix} A & -N \\ -Q & -A^T \end{bmatrix} \in \mathbf{R}^{2n \times 2n}.$$
(8)

If one denotes the transition matrix for G as

$$\Gamma(t,\theta) = \begin{bmatrix} \Gamma_{11}(t,\theta) & \Gamma_{12}(t,\theta) \\ \Gamma_{21}(t,\theta) & \Gamma_{22}(t,\theta) \end{bmatrix} \in \mathbf{R}^{2n \times 2n}, \ \Gamma_{ij}(t,\theta) \in \mathbf{R}^{n \times n}, \ i,j = 1,2,$$
(9)

the solution to the system (7) is

$$\gamma(t) = \Gamma(t, \theta) \gamma(\theta) . \tag{10}$$

Depending on the terminal conditions of the problem, one will adopt $\theta = t_0$ or $\theta = t_f$.

Remark 1: All presented equations are valid for both **P1** and **P2** problems, since no reference to terminal conditions were used.

Remark 2: There are procedures for **P1** problem based on the direct using of the solution (10) to the equation (7), but this solution has a complicated form, especially for feedback control. For the **P2** problem, the equations (5) and (6) are always used.

The same RMDE appears in both problems, the difference consisting in the terminal conditions for this equation. A supplementary difficulty appears in the **P2** problem case since RMDE must be solved in inverse time, starting from

$$\tilde{R}(t_f) = S . \tag{11}$$

This implies to beforehand compute and memorize the solution to RMDE at every sampling moment, or to take again the inverse time computing at every sampling period. According to these procedures, it is not possible to establish a direct time iterative method for RMDE solving.

The present paper establishes similar analytical formulae for RMDE solution in **P1** and **P2** problems. Moreover, one can find the initial value $\tilde{R}(t_0)$ in the **P2** problem and this fact allows establishing of an analytical formula which starts from the initial moment t_0 , or allows the introducing of a direct time numerical iterative techniques.

The Theorem 1 presented in the next section establishes an analytical formula for the solution to RMDE starting from any positive defined initial matrix $\tilde{R}(t_0)$. This initial matrix can be computed with the formulae indicated in the Theorem 2 and 3 for **P1** and **P2** problems, respectively. Finally, the Theorem 4 presents a solution to RMDE in **P2** problems which does not explicitly uses $\tilde{R}(t_0)$.

3. SOLUTION TO RMDE

3.1. Transformation of the canonical equations

The solution (10) to the system (7) is not adequate for analytical developments, since the matrix blocks Γ_{ij} , i,j=1,2, of the transition matrix Γ can not be analytical computed. In order to overcome this difficulty, a change of variables is performed, so that a convenient form for the system matrix is obtained. For this purpose, the co-state variable $\lambda(t)$ is expressed in the form

$$\lambda(t) = Rx(t) + v(t) \tag{12}$$

instead of (4). In the above relation, R and v(t) are unknown constant matrix and variable vector, respectively. One can now introduce the vector

$$\rho(t) = [x(t)^T \quad v(t)^T]^T \in \mathbf{R}^{2n} .$$
(13)

The system for new variables becomes

$$\dot{\rho}(t) = H\rho(t) , \qquad (14)$$

with

$$H = \begin{bmatrix} F & -N \\ H_{21} & -F^T \end{bmatrix} \in \mathbf{R}^{2n \times 2n}, \text{ with } F = A - NR$$
(15)

It is easy to verify that the matrix block H_{21} is zero if the matrix *R* satisfies the equation

$$RNR - R^T A - A^T R - Q = 0 \tag{16}$$

and therefore

$$H = \begin{bmatrix} F & -N \\ 0 & -F^T \end{bmatrix}.$$
 (17)

Evidently, (16) is the Riccati matrix algebraic equation, which appears in the similar optimal control problem with infinite final time (Athans & Falb 1966).

The solution to the equation (14) is

$$\rho(t) = \Omega(t,\theta)\rho(\theta), \quad \Omega(t,\theta) \in \mathbf{R}^{2n \times 2n}, \quad (18)$$

where $\Omega(t,\theta)$ is the transition matrix for *H*. Taking into account the form (17) for *H* and the relationships

 $\dot{\Omega}(t,\theta) = G\Omega(t,\theta)$ and $\Omega(\theta,\theta) = I_{2n}$, one can prove that

$$\Omega(t,\theta) = \begin{bmatrix} \Psi(t,\theta) & \Omega_{12}(t,\theta) \\ 0 & \Phi(t,\theta) \end{bmatrix},$$
(19)

where $\Psi(.)$ and $\Phi(.)$ are the transition matrices for *F* and $-F^T$, respectively and $\Omega_{12}(.)$ satisfies the equation

$$\dot{\Omega}_{12}(t,\theta) = F\Omega_{12}(t,\theta) - N\Phi(t,\theta), \quad \Omega_{12}(\theta,\theta) = 0 \ (20)$$

and it is

$$\Omega_{12}(t,\theta) = \int_{t}^{\theta} \Psi(t,\tau) N \Phi(\tau,\theta) d\tau .$$
 (21)

In this way, the solution to the system (17) can be established using only *n* x *n* transition matrices. This is an advantage in comparison with the well known procedures based on the factorization of the solution to the RMDE and derived from Radon's Lemma (Lainiotis 1976) which use 2nx2n transition matrices. The advantage results from the particular form (17) of the matrix *H* of the new system (14). The matrix blocks of the matrix $\Omega(t, \theta)$ can be computed significantly easier than ones of the matrix $\Gamma(t, \theta)$. In addition, the matrix blocks of the initial transition matrix $\Gamma(t, \theta)$ can be expressed in terms of $\Omega(t, \theta)$; from above transformations, one can prove that

$$\Gamma_{11}(\cdot) = \Psi(\cdot) - \Omega_{12}(\cdot)R, \qquad \Gamma_{12}(\cdot) = \Omega_{12}(\cdot)$$

$$\Gamma_{12}(\cdot) = R\Psi(\cdot) - R\Omega_{12}(\cdot)R - \Phi(\cdot)R, \quad \Gamma_{22}(\cdot) = R\Omega_{12}(\cdot) + \Phi(\cdot)$$
(22)

3.2 Solutions to RMDE in P1 and P2 problems

We are now in position to formulate a general theorem for the solution to RMDE starting from a certain initialization $\tilde{R}(t_0)$. The result can be used for **P1** and **P2** problems adopting adequate transformations.

Theorem 1: The solution to RMDE (6) with initial condition $\tilde{R}(t_0)$ is

$$\tilde{R}(t) = R + \Phi(t, t_0) (\tilde{R}(t_0) - R) M^{-1}(t, t_0) , \qquad (23)$$

where

or

$$M(t,t_0) = \Psi(t,t_0) + \Omega_{12}(t,t_0) [\tilde{R}(t_0) - R].$$
 (24)

Proof: The initial co-state vector $\lambda^0 = \lambda(t_0)$ can be expressed from (4) and (12), written for $t = t_0$:

$$\lambda^0 = \tilde{R}(t_0) x^0, \ x^0 = x(t_0),$$
(25)

$$\lambda^0 = v^0 + Rx^0, \quad x^0 = x(t_0)$$
 (26)

and thus
$$v^0 = [\tilde{R}(t_0) - R] x^0$$
. (27)

Thus, it results from the first equation in (18), written for $\theta = t_0$,

$$x(t) = M(t, t_0)x^0$$
, (28)

with $M(t,t_0)$ given by (24).

The solution given by the second equation (18) is

$$v(t) = \Phi(t, t_0)v^0$$
 (29)

and replacing (27), it results

$$v(t) = \Phi(t, t_0)(\hat{R}(t_0) - R)x^0.$$
(30)

Therefore, the co-state vector can be expressed from (12):

$$\lambda(t) = Rx(t) + \Phi(t, t_0)(R(t_0) - R)x^0.$$
(31)

From (4), (28) and (31), yields (23)

The next two theorems establish the initial matrix $\hat{R}(t_0)$ in **P1** and **P2** problems.

Theorem 2: The initial matrix $\tilde{R}(t_0)$ for RMDE (6) in **P1** problem is

$$\tilde{R}(t_0) = R + \Omega_{12}^{-1}(t_f, t_0) \Psi(t_f, t_0) .$$
(32)

Proof: The first equation (10) for $t = t_f$ and $\theta = t_0$ is

$$\lambda^{0} = \lambda(t_{0}) = -\Gamma_{12}^{-1}(t_{f}, t_{0})\Gamma_{11}(t_{f}, t_{0})x^{0}.$$
 (33)

Since $x(t_f) = 0$ in **P1** problem, it results

$$\lambda^{0} = \lambda(t_{0}) = -\Gamma_{12}^{-1}(t_{f}, t_{0})\Gamma_{11}(t_{f}, t_{0})x^{0}.$$
(34)

One can prove (Botan, Ostafi & Onea 2003) that $\Gamma_{12}(t_f, t_0)$ is non-singular if (A,B) is completely controllable. Comparing (34) with (4) for $t = t_0$, yields

$$\tilde{R}(t_0) = -\Gamma_{12}^{-1}(t_f, t_0)\Gamma_{11}(t_f, t_0).$$
(35)

Replacing the matrices from (22) in (35), it results (23) \blacksquare

Theorem 3: The initial matrix $\tilde{R}(t_0)$ for RMDE (6) in **P2** problem is

$$\tilde{R}(t_0) = R + \Phi(t_0, t_f)(S - R)M_2^{-1}(t_0, t_f).$$
(36)

Proof: The equation (18) written for $\theta = t_f$ is

$$x(t) = \Psi(t, t_f) x(t_f) + \Omega_{12}(t, t_f) v(t_f) .$$
 (37)

One can express $v(t_f)$ from the final condition (11) and from (12) written for $t = t_f$:

$$v(t_f) = (S - R)x(t_f)$$
. (38)

From (37) and (38), one obtains

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$$x(t) = M_2(t, t_f) x(t_f),$$
(39)

$$M_2(t,t_f) = \Psi(t,t_f) + \Omega_{12}(t,t_f)(S-R).$$
(40)

The vector v(t) can be expressed in terms of $x(t_f)$ from the second equation (18) and from (38):

$$v(t) = \Phi(t, t_f)v(t_f) = \Phi(t, t_f)(S - R)v(t)x(t_f)$$
(41)

and then it is related to x(t) using (39). Now, (12) can be written as

$$\lambda(t) = \left[R + \Phi(t, t_f) (S - R) M_2^{-1}(t, t_f) \right] x(t) .$$
 (42)

The matrix $M_2(t,t_f)$ is non-singular since it express the transition between $x(t_f)$ and x(t). Comparing equation (42) with (4), one obtains

$$\tilde{R}(t) = R + \Phi(t, t_f)(S - R)M_2^{-1}(t, t_f) .$$
(43)

Relation (36) results immediately from (43) for $t = t_0 \blacksquare$

Remark 3: Relation (43) represents an analytical formula for the solution to RMDE for **P2** problem. This formula is also indicated by Botan and Ostafi (2010). A solution in a closely form with (43) is proved by Rusnak (1988) by straightforward computing. The formula implies only *nxn* transition matrices and this fact is an advantage in comparison with other similar analytical methods which imply the computing of 2nx2n transition matrices. As in other methods, t_f is reference moment, but anyway, the use of the formula (43) has significant advantages from the simplification and the precision point of view.

Remark 4: The results of the Theorem 1, combined with Theorem 2 or 3 allow the analytical computation in direct time of the solution to RMDE for the **P1** and **P2** problems, respectively. The formula (23) from the first theorem can be used as such, but it is possible to introduce in addition certain iterative computing. For instance, the transition matrices can be computed with

$$\begin{split} \Psi_{i+1} &= \Psi(t_{i+1}, t_0) = \Psi_i \cdot \Psi_{\delta}, \ \Psi_0 = I \\ \Phi_{i+1} &= \Phi(t_{i+1}, t_0) = \Phi_i \cdot \Phi_{\delta}, \ \Phi_0 = I \,, \end{split}$$

where $\Psi_{\delta} = \Psi(t_{i+1}, t_i)$ and similar for Φ_{δ} . The iterative computing can be also performed for the matrix $\Omega_{12}(.)$ starting from (20):

$$\Omega_{12}(t_{i+1},t_0) = (I - \delta F) \Omega_{12}(t_i,t_0) - \delta N \Phi(t_i,t_0) ,$$

with $\delta = t_{i+1} - t_i$ and $\Omega_{12}(t_0, t_0) = 0$.

Remark 5: Different numerical iterative (in direct time) procedures can be used for both problems, starting from the known initial matrix $\tilde{R}(t_0)$. For instance, a simple way is to approximate the derivative in (6) and then one obtain

$$\tilde{R}_{i+1} = \tilde{R}_i + \delta(\tilde{R}_i N \tilde{R}_i - \tilde{R}_i A - A^T \tilde{R}_i - Q);
\delta = t_{i+1} - t_i = const.,$$
(44)

where $\tilde{R}_i = \tilde{R}(t_i)$ and the initialization is $\tilde{R}_0 = \tilde{R}(t_0)$, specific for each problem. One has to remark that some procedures of this type (for instance, (44)) impose a great number of steps for an adequate accuracy.

The Theorems 1 and 3 offer the solution to **P2** problem explicitly computing the initial matrix $\tilde{R}(t_0)$. The next theorem indicates an analytical formula for the solution to RMDE for **P2** problem which does not involve $\tilde{R}(t_0)$.

Theorem 4: The solution to RMDE (6) with final condition (11) is

$$\tilde{R}(t) = R + \Phi(t, t_0) W M^{-1}(t, t_0) , \qquad (45)$$

with $M(t,t_0)$, $M_2(t,t_f)$ given by (24) and (40) and

$$W = \Phi(t_0, t_f)(S - R)M_2^{-1}(t_0, t_f).$$
(46)

Proof: the result is obtained immediately form Theorems 1 and 3. With the adopted notation, (36) can be written

$$\tilde{R}(t_0) - R = W . \tag{47}$$

If this relation is replaced in (23), yields (45). ■

Remark 6: The formulae given by the above theorems are similar to some extent using only nxn transition matrix. They are solution to the same RMDE, but for different terminal conditions. The formulated problems have unique solutions (supplementary controllability condition is necessary for **P1** problems). For this reason, all different methods have to lead to the same result.

Remark 7: Similar results can be obtained for other problem types, for example, for discrete time LQ problems. The established formulae can also be extended to the time variant case. For instance, instead of (43) from Theorem 2, the solution to RMDE is

$$\tilde{R}(t) = \bar{R}(t) + \Phi(t, t_f)(S - \bar{S})M_2^{-1}(t, t_f), \qquad (48)$$

where $\overline{R}(t)$ is a particular solution of RMDE with final condition $\overline{R}(t_f) = \overline{S}$. It should be noted that the advantages for the time-variant case can be achieved if the finding of a particular solution is not difficult.

Remark 8: Since $\hat{R}(t)$ has two components (see, for instance (43) or (45)), the optimal control will also contain two components, which can be expressed from (3) and (12) in the form

$$u(t) = u_f(t) + u_c(t), \qquad (49)$$

where

$$u_f(t) = -P^{-1}B^T R x(t)$$

is a feedback component (identical with the control vector in the similar LQ problem with infinite final time) and

$$u_{c}(t) = -P^{-1}B^{T}v(t) = -P^{-1}B^{T}\Phi(t,t_{0})v(t_{0})$$

is a corrective component.

The difference between the **P1** and **P2** problems consists in the initialization $v(t_0)$. The procedures of this type will be named in the sequel as first procedure for optimal controller implementation (FP). Similarly, the procedures based on (5) and above established solutions for RMDE will be named as second procedures (SP).

4. EXAMPLES

A first category of examples refers to the computing of the solution to RMDE for **P2** problem. Different examples of systems (from second to ten order) were considered and in

each case were used three computing methods: two methods are based on the analytical formulae (43) and (45) from the above theorems; the third one uses a numerical iterative technique, starting from the initial value (for instance, based on (44)). A first comparison refers to the precision of the results. The indices $\rho_2 = \|\tilde{R}_1(t) - \tilde{R}_2(t)\| / \|\tilde{R}_1(t)\|$ and $\rho_3 = \|\tilde{R}_1(t) - \tilde{R}_3(t)\| / \|\tilde{R}_1(t)\|$ were introduced for this purpose. In these ratios, $\tilde{R}_1(t), \tilde{R}_2(t), \tilde{R}_3(t)$ denote the solutions to RMDE obtained with the above mentioned methods (equations (43), (45) and (44), respectively). The maximal values for ρ_2 and ρ_3 for all exemplified cases and for all t were $6 \cdot 10^{-13}$ and $3 \cdot 10^{-3}$, respectively. This fact indicates a very high precision for analytical methods and an acceptable precision for the adopted numerical method. The value of ρ_3 depends on the increment δ . For instance, one obtained $\rho_3 = 3.2 \cdot 10^{-2}$ for $\delta = T/10$ (T is the sampling period) and $\rho_3 = 0.63 \cdot 10^{-2}$ for $\delta = T / 50$ in a second order system case.

Besides precision, the computing time was analyzed. For this aim, adequate MATLAB functions (TIC and TOC) were used in order to appreciate the total computing time of the optimal control on the interval $[t_0, t_f]$. Close values were obtained for the two analytical methods (based on (43) and (45)) and a value three times greater for the numerical procedure (based on (44)) for $\delta = T/50$. By comparison, this last case offers a value twelve times better as the classical procedure, based on the same iterative formula (44), but used in inverse time, starting from $\tilde{R}(t_f)$.

In conclusion, the proposed analytical methods offer advantages relating to the precision and computing time in comparison with classical iterative procedures. An intermediate behaviour from the computing time point of view appears to the numerical iterative procedures based on analytical formulae.

A second category of examples and simulation refers to the behaviour of the optimal control for both **P1** and **P2** problems. The Fig. 1 and Fig. 2 present the variation of the optimal control and state variables for the **P1** and **P2** problems, respectively.

The simulations were performed for the system (1) and the criterion (2) with the matrices:

$$A = \begin{bmatrix} -0.04 & 20 \\ -3.5 & -19 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, P = 1.$$

In both figure, the results based on two methods are presented: FP (line curves) and SP (x-mark curve) – see Remark 8. The explanation of the very good coincidence is the fact that the both methods are in essence analytical and only small numerical computing errors occur. One can remark that the components of the corrective vector v(t) have about similar variations, but they start from other initial values in the two cases.



Fig. 1. Behaviour of the optimal system – fixed end-point case (for FP and SP)



Fig. 2. Behaviour of the optimal system – free end-point case (for FP and SP)

5. CONCLUSIONS

- Certain new solutions to Riccati differential equations in the LQ problems are proposed.

- A unified procedure for the problems with free and fixed end-point is approached.

- A Riccati equation with adequate initial condition and the corresponding analytical solution are established for LQ problems with fixed end point.

- Two analytical solutions to Riccati equation for the LQ problem with free end-point are proposed. One of them starts from an initial condition, avoiding the inverse time computing. A recurrent numerical method for direct time computing of the solution to Riccati equation is also indicated.

- All the proposed methods ensure a high precision, a significant decrease of the computing time and an easier implementation in comparison with the existing procedures.

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