Stabilization of discrete-time hybrid systems with time-delays

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Abstract: In this paper the problem of stabilization of a particular class of hybrid systems is approached. For the class of discrete-time hybrid systems with time-delays, it is shown that under some assumption it is possible to express these systems in a common representation: piecewise affine systems. The stability of such systems is proved through Lyapunov theory, more accurate through quadratic, multiple Lyapunov functions. For stabilization task we design a piecewise linear state feedback control law. Finally, some simulation results are presented.

Keywords: discrete-time hybrid systems with time-delays, piecewise affine systems, multiple Lyapunov functions.

1. INTRODUCTION

Hybrid systems are dynamical systems with two components: discrete and continuous dynamics. The discrete dynamics are represented by digital automaton or algebraic equations, and the continuous dynamics are represented by differential or difference equations. Generally, the study of the hybrid systems is defined as the study of systems which involve interaction of the above dynamics.

In the last years, the problems of analysis, verification, computation, stability and stabilization of hybrid system were studied with growing interest.

In this paper, we deal with stability and stabilization of a particular class of hybrid systems, i.e. discrete-time hybrid systems with time-delays.

The main approaches existing in literature for stability of hybrid systems are related to Lyapunov stability and Lagrange stability (Davrazos and Koussoulas, 2001). The Lyapunov technique uses four concepts to prove stability of the studied systems: common Lyapunov functions, multiple Lyapunov functions, modified Lyapunov functions and Poincaré maps. For discrete-time affine systems the main approaches are based on common and multiple Lyapunov functions (Ferrari *et al*., 2001).

The paper is organized as follows: in Section 2 is defined a new class of hybrid systems, i.e. discrete-time hybrid systems with time-delays, in Section 3 is presented a method to transform this class of hybrid system in a common representation such as, piecewise affine systems (Ferrari *et al*., 2001), in Sections 4 and 5 is presented an illustrative example for which we studied the stability and stabilization.

2. DISCRETE-TIME HYBRID SYSTEMS WITH TIME-DELAYS

The hybrid systems with time-delays represent a new research direction in modelling and control areas (Kulkarni *et al*., 2007; Liu, 2010).

Definition 1: A general discrete-time hybrid system with time-delays is an invariant, linear, discrete-time system defined by the following family of piecewise affine systems with time-delays:

$$\begin{cases} x(k+1) = A_i x(k) + \sum_{v=1}^{V} A_{iv} x(k-\tau_v) + B_i u(k) + \\ \qquad + \sum_{r=1}^{R} B_{ir} u(k-\varphi_r) + f_i \\ y(k) = C_i x(k) + \sum_{l=1}^{L} C_{il} x(k-\psi_l) + D_i u(k) + \\ \qquad + \sum_{w=1}^{W} D_{iw} u(k-\eta_w) + g_i \end{cases} \quad (1)$$

$$if \quad \left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in X_i \right)$$

where: $x(k) \in \Re^n, u(k) \in \Re^m$ and $y(k) \in \Re^p$ are the state, input and output at time k, $A_i, A_{iv} \in \Re^{n\times n}$, $B_i, B_{ir} \in \Re^{n\times m}$, $C_i, C_{il} \in \Re^{p\times n}$ and $D_i, D_{iw} \in \Re^{p\times m}$ are the appropriate size system matrices, $\tau_v\ \left(v=\overline{1,V}\right), \varphi_r\ \left(r=\overline{1,R}\right), \psi_l\ \left(l=\overline{1,L}\right), \eta_w\ \left(w=\overline{1,W}\right) \in N^*$ are time-delays, $V, R, L, W \in N$, f_i and g_i are real vectors, $X_i = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : S_i x + R_i u \le T_i \right\}, i = 1 \ldots s$ are polyhedral partitions in state-input space.

Definition 2: A discrete-time, invariant, linear, hybrid system with a constant state time-delay is a subclass of a family of system (1), described by the following family of piecewise affine systems with time-delay:

$$\begin{cases} x(k+1) = A_i x(k) + A_{iv} x(k-h) + B_i u(k) + f_i \\ y(k) = C_i x(k) + C_{il} x(k-h) + D_i u(k) + g_i \end{cases}$$
$$if \quad \left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in X_i \right) \tag{2}$$

where h is constant time-delay.

Definition 1.3: A discrete-time, invariant, linear, hybrid system with a constant input time-delay is a subclass of a family of system (1), described by the following family of piecewise affine systems with time-delay:

$$\begin{cases} x(k+1) = A_i x(k) + B_i u(k) + B_{ir} u(k-h) + f_i \\ y(k) = C_i x(k) + D_i u(k) + D_{iw} u(k-h) + g_i \end{cases}$$
$$if \quad \left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in X_i \right) \tag{3}$$

where h is constant time-delay.

3. EQUIVALENCE OF TWO CLASSES OF HYBRID SYSTEMS

Obviously, the representations (2) and (3) of the hybrid system can be translated using appropriate state transformation in a common representation such as, piecewise affine systems.

We consider the following transformation:

$$\left\{ z_j(k) \overset{\Delta}{=} x(k-j), \qquad j = 1 \ldots h \right. \tag{4}$$

then, the system (2) becomes:

$$\begin{cases} \bar{x}(k+1) = \bar{A}_i \bar{x}(k) + \bar{B}_i u(k) + \bar{f}_i \\ y(k) = \bar{C}_i \bar{x}(k) + D_i u(k) + g_i \end{cases}$$
$$if \quad \left(\begin{bmatrix} M\bar{x}(k) \\ u(k) \end{bmatrix} \in X_i \right) \tag{5}$$

where:

$$\bar{x} = \begin{bmatrix} x^T(k) & z_h^T(k) & z_{h-1}^T(k) & \cdots & z_1^T(k) \end{bmatrix}^T,$$
$$\bar{f}_i = \begin{bmatrix} f_i^T & 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$
$$\bar{A}_i = \begin{bmatrix} A_i & A_{i1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$
$$\bar{C}_i = \begin{bmatrix} C_i & C_{i1} & 0 & 0 & \cdots & 0 \end{bmatrix}, M = \begin{bmatrix} I_{n\times n} & O_{n\times hn} \end{bmatrix},$$

where $O_{n\times hn}$ is a $n\times hn$ zero matrix and $I_{n\times n}$ is a $n\times n$ identity matrix.

Similarly with the above procedure, we consider the following transformation:

$$\left\{ v_j(k) \overset{\Delta}{=} u(k-j), \qquad j = 1 \ldots h \right. \tag{6}$$

then the system (3) is transformed in (5) with the following parameters:

$$\bar{x} = \begin{bmatrix} x^T(k) & v_h^T(k) & v_{h-1}^T(k) & \cdots & v_1^T(k) \end{bmatrix}^T,$$
$$\bar{f}_i = \begin{bmatrix} f_i^T & 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$
$$\bar{A}_i = \begin{bmatrix} A_i & B_{i1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \\ 0 \\ \vdots \\ I \end{bmatrix},$$
$$\bar{C}_i = \begin{bmatrix} C_i & D_{i1} & 0 & 0 & \cdots & 0 \end{bmatrix}, M = \begin{bmatrix} I_{n\times n} & O_{n\times hm} \end{bmatrix}$$

In Heemels *et al*. (2001), under additional assumptions, it is shown that the equivalences between several subclasses of hybrid systems are obtained. Some examples of such subclasses are: linear complementary systems (Heemels *et al*., 1999), mixed logical dynamical systems (Bempoard and Morari, 1999), piecewise affine systems (Ferrari *et al*., 2001).

4. STABILITY OF A DISCRETE-TIME HYBRID SYSTEM WITH STATE TIME-DELAY

We consider a system of form (2) with following parameters:

$$i = 1, 2; \quad h = 2; \quad x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T; u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T;$$
$$A_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix}; A_2 = \begin{bmatrix} -0.5 & 0.3 \\ 0.5 & -0.1 \end{bmatrix}$$
$$A_{1v} = \begin{bmatrix} -0.6 & -0.1 \\ -0.1 & -0.2 \end{bmatrix}; A_{2v} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; B_2 = \begin{bmatrix} 5 & 0 \\ 5 & -1 \end{bmatrix}, C_1 = C_2 = I_{2\times 2}; \tag{7}$$
$$C_{1l} = C_{2l} = O_{2\times 2}; D_1 = D_2 = O_{2\times 2},$$
$$f_1 = f_2 = g_1 = g_2 = O_{2\times 1}$$
$$\begin{cases} [x_1(k)\, x_2(k)]^T \in X_1 & if \quad x_1(k) - x_2(k) \le 0 \\ [x_1(k) \;\; x_2(k)]^T \in X_2 & if \quad x_1(k) - x_2(k) > 0 \end{cases}$$

Using the state transformation presented in Section III, the system is transformed into common representation (5):

$$\begin{cases} \bar{x} = [x_1 \quad x_2 \quad z_{21} \quad z_{22} \quad z_{11} \quad z_{12}]^T; \\ z_{21}(k) = x_1(k-2),\, z_{22}(k) = x_2(k-2), \\ z_{11}(k) = x_1(k-1),\, z_{12}(k) = x_2(k-1), \\ \bar{C}_1 = \bar{C}_2 = [I_{2\times 2} \quad O_{2\times 4}],\, \bar{f}_1 = \bar{f}_2 = O_{6\times 1} \end{cases} \tag{8a}$$

$$\bar{A}_1 = \begin{bmatrix} -0.2 & 0.1 & -0.6 & -0.1 & 0 & 0 \\ 0 & 0.3 & -0.1 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},\ \bar{B}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ \multicolumn{2}{c}{O_{4\times 2}} \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} -0.5 & 0.3 & 0.2 & 0 & 0 & 0 \\ 0.5 & -0.1 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},\ \bar{B}_2 = \begin{bmatrix} 5 & 0 \\ 5 & -1 \\ \multicolumn{2}{c}{O_{4\times 2}} \end{bmatrix} \tag{8b}$$

$$\begin{cases} [M\bar{x}(k)] \in X_1 \quad if \quad N\bar{x}(k) \le 0 \\ [M\bar{x}(k)] \in X_2 \quad if \quad N\bar{x}(k) > 0 \end{cases}$$

where $N = [1 \quad -1 \quad O_{1\times 4}]$ and $M = [I_{2\times 2} \quad O_{2\times 4}]$

In figure 1 is presented the free evolution of states of discrete-time hybrid system with state delay (evolution of the first two states of piecewise affine system). In figure 2 are presented the active partitions of the piecewise affine system. The evolutions are obtained for the following initial conditions:

$$x_1(0) = z_{21}(0) = z_{11}(0) = 2,\ x_2(0) = z_{22}(0) = z_{12}(0) = 4$$
$$u_1(k) = u_2(k) = 0,\ 0 \le k \le 100$$

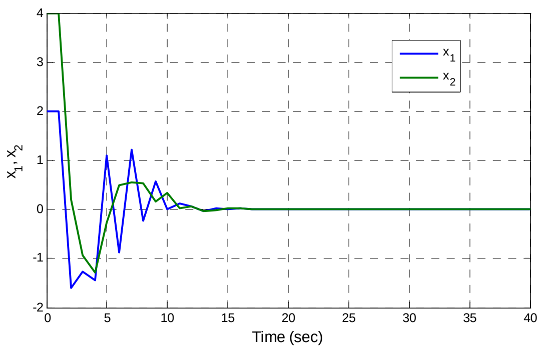

Fig. 1. Evolution of states of discrete-time hybrid system with state delay.

The stability of piecewise affine systems was investigated in several papers (Johansson, 2007; Ferrari *et al.*, 2001; Mignone, 2002). The main approaches use the common or multiple quadratic Lyapunov functions. Based on the results presented in the above references, we can establish the following result:

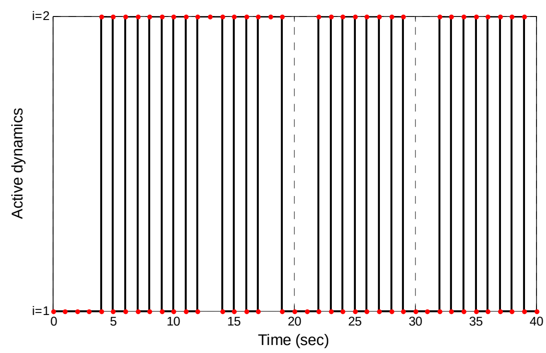

Fig. 2. The active dynamics of the piecewise affine system.

Proposition 1: The origin of the system $\bar{x}(k+1) = \bar{A}_i \bar{x}(k)$, where $\bar{A}_i,\ i = 1, 2$ are given by (8), is asymptotically stable, if there exist matrices $P_i = P_i^T > 0,\quad i = \{1, 2\}$ such that the positive definite function $V = (\bar{x})^T P_i \bar{x},\quad \forall \bar{x} \in X_i$, satisfies:

$$V(\bar{x}(k)) - V(\bar{x}(k-1)) < 0,\ \forall k \in N^* \tag{9}$$

The quadratic, multiple Lyapunov functions can be computed by solving the LMIs:

$$\begin{aligned} & P_1 = P_1^T > 0, \quad P_2 = P_2^T > 0 \\ & \bar{A}_1^T P_2 \bar{A}_1 - P_1 < 0, \quad \bar{A}_2^T P_1 \bar{A}_2 - P_2 < 0 \\ & \bar{A}_1^T P_1 \bar{A}_1 - P_1 < 0, \quad \bar{A}_2^T P_2 \bar{A}_2 - P_2 < 0 \end{aligned} \tag{10}$$

Solving system (10) we determine the matrices P_1 and P_2, so:

$$P_1 = \begin{bmatrix} 48.38 & 0.90 & 0.08 & 1.70 & 6.39 & -2.93 \\ 0.90 & 53.72 & -1.35 & -2.65 & -3.69 & -1.31 \\ 0.08 & -1.35 & 26.95 & 2.45 & 4.09 & -2.39 \\ 1.70 & -2.65 & 2.45 & 19.76 & -0.96 & -0.29 \\ 6.39 & -3.69 & 4.09 & -0.96 & 35.39 & 1.55 \\ -2.93 & -1.31 & -2.39 & -0.29 & 1.55 & 33.53 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 63.72 & -4.94 & -4.95 & 2.53 & 7.97 & -4.33 \\ -4.94 & 52.89 & 2.31 & -0.87 & -4.87 & 0.01 \\ -4.95 & 2.31 & 20.17 & 0.76 & 1.91 & -1.33 \\ 2.53 & -0.87 & 0.76 & 19.05 & -1.11 & -0.56 \\ 7.97 & -4.87 & 1.91 & -1.11 & 34.48 & 1.89 \\ -4.33 & 0.01 & -1.33 & -0.56 & 1.89 & 33.32 \end{bmatrix} \tag{11}$$

The quadratic multiple Lyapunov functions are expressed by:

$$V_i(\bar{x}) = (\bar{x})^T P_i \bar{x}, \quad \forall i = 1, 2 \tag{12}$$

In figures 3 and 4 are represented the quadratic Lyapunov functions for considered subsystems ($i = 1$ and $i = 2$).

We observe that every moment k when the subsystem i became active, the corresponding energy function V_i is decreasing from the value V_i which had the last time when the subsystem i was active.

Form figure 5, where is represented $V(\bar{x})$, it can be observed that the inequality (9) is satisfied for all $k \in N^*$.

5. STABILIZATION OF A DISCRETE-TIME HYBRID SYSTEM WITH STATE TIME-DELAY

For hybrid systems of form (5), with parameters given by (7) and (8) we can determine a piecewise linear state feedback control law of form:

$$u(k) = K_i \bar{x}(k), \quad \forall i = 1, 2 \quad \forall k \tag{13}$$

where $K_i = W_i P_i, \forall i = 1, 2$, and W_i are solutions of the following LMIs (Mignone, 2002):

$$\begin{matrix} \begin{pmatrix} Q_1 & (A_2 Q_2 + B_2 W_2) \\ (A_2 Q_2 + B_2 W_2)^T & Q_2 \end{pmatrix} > 0 \\ \begin{pmatrix} Q_2 & (A_1 Q_1 + B_1 W_1) \\ (A_1 Q_1 + B_1 W_1)^T & Q_1 \end{pmatrix} > 0 \end{matrix} \tag{14}$$

where $Q_1 = P_1^{-1}, Q_2 = P_2^{-1}$, and P_1, P_2 are given by (11).

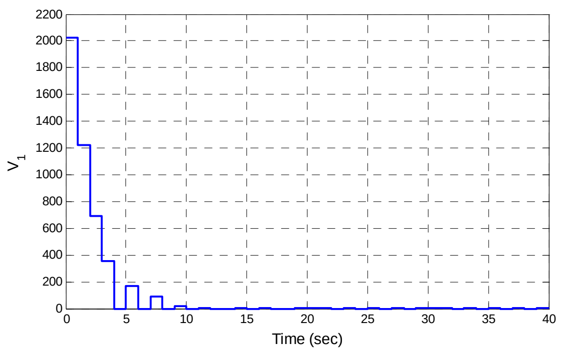

Fig. 3. The quadratic Lyapunov function V_1.

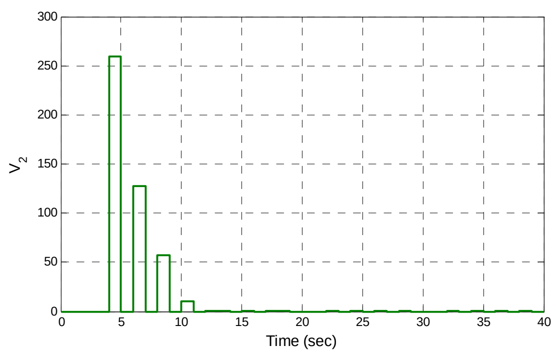

Fig. 4. The quadratic Lyapunov function V_2.

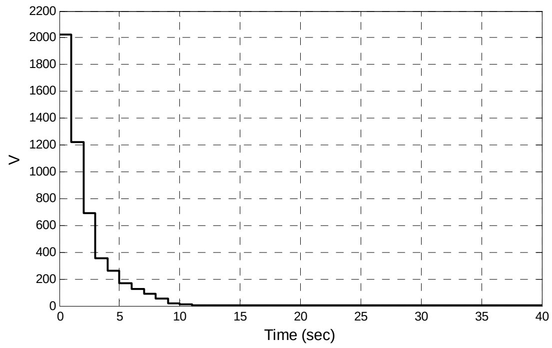

Fig. 5. The quadratic Lyapunov function V.

Solving the system (14) we determine the solutions:

$$W_1 = \begin{bmatrix} 0.0023 & 0 \\ -0.0007 & -0.0017 \\ 0.0114 & 0.0009 \\ 0.0007 & 0.003 \\ -0.0018 & -0.0002 \\ 0.0011 & 0 \end{bmatrix}^T$$

$$W_2 = \begin{bmatrix} 0.0014 & 0.0154 \\ -0.001 & -0.0062 \\ -0.0015 & -0.005 \\ -0.0002 & 0.0029 \\ -0.0004 & -0.0042 \\ 0.0002 & 0.0021 \end{bmatrix}^T$$

So, the control law (13) is expressed by:

$$\begin{array}{l} \begin{cases} u_1(k) = [0.1 \quad -0.05 \quad 0.3 \quad 0.05 \quad 0 \quad 0] \cdot \bar{x}(k) \\ u_2(k) = [0 \quad -0.1 \quad 0.033 \quad 0.066 \quad 0 \quad 0] \cdot \bar{x}(k) \end{cases} \\ if \quad N\bar{x}(k) \le 0 \\ \begin{cases} u_1(k) = [0.1 \quad -0.06 \quad -0.04 \quad 0 \quad 0 \quad 0] \cdot \bar{x}(k) \\ u_2(k) = [1 \quad -0.4 \quad -0.2 \quad 0.1 \quad 0 \quad 0] \cdot \bar{x}(k) \end{cases} \\ if \quad N\bar{x}(k) > 0 \end{array} \tag{15}$$

In figures 6 and 7 is represented the evolution of the first two states of piecewise affine system in closed loop with control law (15), respectively the active dynamics of the system.

We observe that the states of the piecewise affine system, and so the states of initial discrete-time hybrid systems converge asymptotically to zero, when $k \to \infty$.

6. CONCLUSIONS

In this paper we presented a new class of hybrid systems, i.e. discrete-time hybrid systems with time-delays. This type of hybrid system, can be translated using a state augmentation method in other common class, for example piecewise affine systems. Based on this transformation, we presented a method to prove stability and stabilization of the piecewise affine system, and finally the initial hybrid system. The stability of the system is proved by

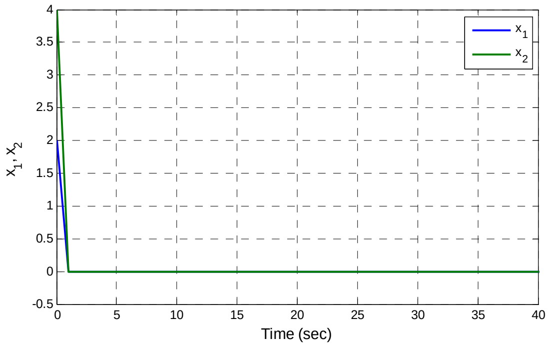

Fig. 6. Evolution of states of piecewise affine system.

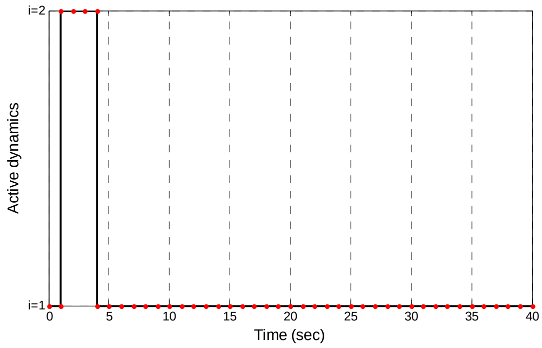

Fig. 7. The active dynamics of the piecewise affine system.

using a quadratic, multiple Lyapunov functions. For stabilization task a piecewise linear state feedback control law was determined. Finally, some simulation results were presented.

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